Problem 4

The figure shows a point P on the parabola $y = x^2$ and the point Q where the perpendicular bisector of OP intersects the y-axis. As P approaches the origin along the parabola, what happens to Q? Does it have a limiting position? If so, find it.

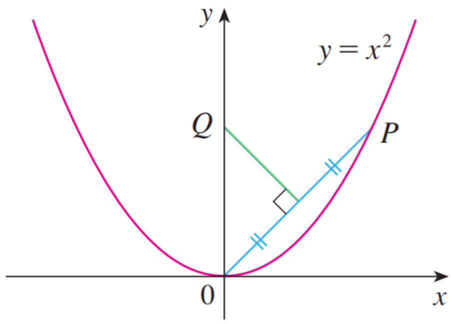
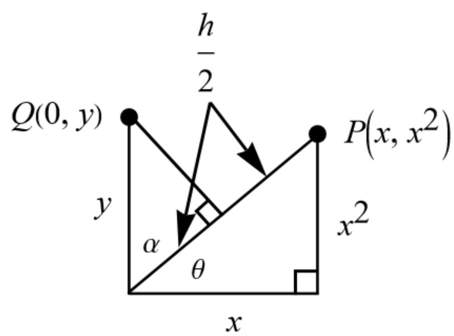


FIGURE FOR PROBLEM 4

Solution

Use geometry and trigonometry to write a formula for the height of Q.



Use the definition of cosine.

$$\cos\alpha = \frac{h/2}{y} = \frac{h}{2y}$$

Solve for the height.

$$y = \frac{h}{2\cos\alpha} \tag{1}$$

Use the definition of tangent.

$$\tan\theta = \frac{x^2}{x}$$

Solve for θ.

$$\theta = \tan^{-1}\left(\frac{x^2}{x}\right)$$

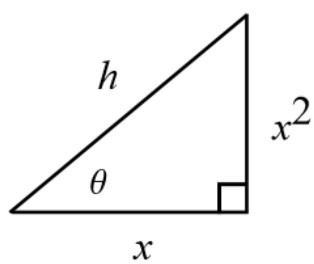
Notice that α and θ add to 90°, or $\pi/2$ radians.

$$\alpha + \theta = \frac{\pi}{2} \quad \rightarrow \quad \alpha = \frac{\pi}{2} - \theta = \frac{\pi}{2} - \tan^{-1}\left(\frac{x^2}{x}\right)$$

As a result, equation (1) becomes

$$\begin{aligned} y &= \frac{h}{2\cos\alpha} \qquad (1) \\ &= \frac{h}{2\cos\left[\frac{\pi}{2} - \tan^{-1}\left(\frac{x^2}{x}\right)\right]} \\ &= \frac{h}{2\left[\cos\frac{\pi}{2}\cos\tan^{-1}\left(\frac{x^2}{x}\right) + \sin\frac{\pi}{2}\sin\tan^{-1}\left(\frac{x^2}{x}\right)\right]} \\ &= \frac{h}{2\left[(0)\cos\tan^{-1}\left(\frac{x^2}{x}\right) + (1)\sin\tan^{-1}\left(\frac{x^2}{x}\right)\right]} \\ &= \frac{h}{2\sin\left[\tan^{-1}\left(\frac{x^2}{x}\right)\right]}. \qquad (2) \end{aligned}$$

Draw the implied right triangle from the angle $\theta = \tan^{-1}\left(\frac{x^2}{x}\right)$.



$$\sin\theta = \frac{x^2}{h}$$

So then equation (2) becomes

$$\begin{aligned} y &= \frac{h}{2\left(\frac{x^2}{h}\right)} \\ &= \frac{h^2}{2x^2}. \end{aligned}$$

Use the Pythagorean theorem to eliminate h in favor of x.

$$\begin{aligned} y &= \frac{(x)^2 + (x^2)^2}{2x^2} \\ &= \frac{x^2 + x^4}{2x^2} \end{aligned}$$

To find out what happens to Q as P approaches the origin along the parabola, take the limit of y as $x \to 0$.

$$\begin{aligned}\lim_{x\to 0} y &= \lim_{x\to 0} \frac{x^2 + x^4}{2x^2} \\ &= \lim_{x\to 0} \left(\frac{x^2}{2x^2} + \frac{x^4}{2x^2}\right) \\ &= \lim_{x\to 0} \left(\frac{1}{2} + \frac{x^2}{2}\right) \\ &= \frac{1}{2}\end{aligned}$$

Therefore, as P approaches the origin along the parabola, Q approaches the point $\left(0, \frac{1}{2}\right)$.